Identifying causal effects from observational data

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Goal

- Estimate the **total causal effect** of $X$ on $Y$ - the change in $Y$ due to $do(x)$ - from observational data.

- $do(x)$: an intervention that sets variables $X$ to $x$. $f(y|do(x)) \neq f(y|x)$.

Observational data

Randomized control studies
Causal Directed Acyclic Graph (DAG) $\mathcal{D}$. 
Interventional Causal DAG

Causal DAG $\mathcal{D}$ after a “do”-intervention on $X_1$. 
Interventional density

- \textit{do(x)}: an intervention that sets variables X to x.
- Observational density \( f(v) \), Interventional density \( f(v|\text{do}(x)) \).

A DAG \( \mathcal{D} \) is causal if for all observational and interventional densities:

\[
\begin{align*}
  f(v) &= \prod_{v_j \in \mathbf{V}} f(v_j|\text{pa}(v_j, \mathcal{D})) \quad \text{and} \quad f(v|\text{do}(x)) = \prod_{v_j \in \mathbf{V}\setminus X} f(v_j|\text{pa}(v_j, \mathcal{D})).
\end{align*}
\]

\[
\begin{align*}
  f(b, x, y) &= f(y|b, x)f(x|b)f(b) & f(b, y|\text{do}(x)) &= f(y|b, x)f(b) \\
  f(b, y|x) &= f(y|b, x)f(b|x) \neq f(b, y|\text{do}(x)).
\end{align*}
\]
How to define a causal effect?

Total causal effect

- Total causal effect - $\tau_{yx}$ - is some functional of $f(y|do(x)), P(Y|do(x))$.
- Examples: $E[Y|do(X = x + 1)] - E[Y|do(X = x)], \frac{\partial}{\partial x} E(Y|do(x))$, OR, RR...

Identifiability

- A causal effect is identifiable from observational data if $f(y|do(x))$ is computable from $f(v)$.
- Given the causal DAG, every total causal effect is identifiable.

$$f(y|do(x)) = \int f(b,y|do(x))db$$

$$= \int f(y|b,x)f(b)db.$$
Problem solved?

DAG $\mathcal{D}$. 
Problem solved?

Completed Partially Directed Acyclic Graph (CPDAG) $C$. 
Completed Partially Directed Acyclic Graph (CPDAG) $C$. 
Problem solved?

Completed Partially Directed Acyclic Graph (CPDAG) $C$. 
Problem solved?

Completed Partially Directed Acyclic Graph (CPDAG) $C$. 
Maximally oriented Partially Directed Acyclic Graph (MPDAG) $\mathcal{G}$.
• PC (Spirtes et al, 1993), GES (Chickering, 2002)
• Adding background knowledge (Meek, 1995; TETRAD, Scheines et al., 1998), PC LINGAM (Hoyer et al., 2008), GIES (Hauser and Bühlmann, 2012), IGSP (Wang et al., 2017), etc.
## Overview of graphical criteria for identification

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<th>DAG</th>
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- ⇒ - sufficient for identification,
- ⇔ - necessary and sufficient for identification

**Adjustment:** Z is an adjustment set if

\[
f(y|do(x)) = \int f(y|x, z)f(z)dz
\]

**G-formula:** Let \( V' = V \setminus (X \cup Y) \), then

\[
f(y|do(x)) = \int \prod_{v_i \in V \setminus X} f(v_i|pa(v_i, D))dV'.
\]
Does an adjustment set always exist?

If \( X = \{X\}, \ Y = \{Y\} \):

**Proposition** (Perković, 2020)

If \( Y \notin Pa(X, G) \), then an adjustment set relative to \( (X, Y) \) exists in the MPDAG \( G \), if and only if the \( f(y|do(x)) \) is identifiable given \( G \).

What about for \(|X| > 1\), or \(|Y| > 1\)?

Does an adjustment set always exist?

**No. Not even in a DAG.**
### Graphical criterion

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⇒ - sufficient for identification,
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Causal identification formula

**Theorem** (Perković, 2020)

If all proper possibly causal paths from $X$ to $Y$ start with a directed edge in $G$, then

$$f(y|do(x)) = \int \prod_{i=1}^{k} f(s_i|pa(s_i,G))ds,$$

where $S = an(Y, G_{V\setminus X}) \setminus Y$, and $(S_1, \ldots, S_k)$ is a partition of $S \cup Y$ into undirected connected sets in $G$.

- $S \cup Y = an(Y, G_{V\setminus X})$ - nodes that have a causal path to $Y$ that is not through $X$.
- $(S_1, \ldots, S_k)$ - maximal connected components of $S \cup Y$ in the induced undirected subgraph of $G$. 
How to use the causal identification formula?

\[
f(y|do(x_1, x_2)) = \int f(y|a, b, c, d, x_2)f(a|b, d, x_1)f(b, c, d)da \, db \, dc \, dd
\]
$f(y|do(x_1, x_2)) =$?

- $S = an(Y, GV \setminus X) \setminus \{Y\} = \{A, B, C, D\}$, Partition of $S \cup \{Y\} =$
\( f(y \mid do(x_1, x_2)) =? \)

\[ S = an(Y, \mathcal{G}_V \setminus X) \setminus \{Y\} = \{A, B, C, D\}, \text{ Partition of } S \cup \{Y\} = (\{B, C, D\}, \{A\}, \{Y\}). \]
\[ f(y \mid \text{do}(x_1, x_2)) =? \]

\[ S = an(Y, G \backslash X) \setminus \{Y\} = \{A, B, C, D\}, \text{ Partition of } S \cup \{Y\} = (\{B, C, D\}, \{A\}, \{Y\}). \]

\[
f(y \mid \text{do}(x_1, x_2)) = \int f(y, s \mid \text{do}(x_1, x_2))ds = \int f(y, a, b, c, d \mid \text{do}(x_1, x_2))ds
\]

\[
= \int f(y \mid a, b, c, d, \text{do}(x_1, x_2)) f(a \mid b, c, d, \text{do}(x_1, x_2)) f(b, c, d \mid \text{do}(x_1, x_2))ds
\]

\[
= \int f(y \mid a, b, c, d, \text{do}(x_1, x_2)) f(a \mid b, d, \text{do}(x_1, x_2)) f(b, c, d \mid \text{do}(x_1, x_2))ds
\]

\[
= \int f(y \mid a, b, c, d, x_2) f(a \mid b, d, x_1) f(b, c, d)ds.
\]
Estimation in the linear case

\[ \begin{align*}
\tau_{yx} &= (\tau_{yx_1.x_2}, \tau_{yx_2.x_1})^T = (\alpha_1 \gamma_1, \alpha_2)^T = \left( \frac{\partial E[Y|do(x_1,x_2)]}{\partial x_1}, \frac{\partial E[Y|do(x_1,x_2)]}{\partial x_2} \right)^T \\
E[Y|do(x_1,x_2)] &= \int y f(y|a, b, c, d, x_2) f(a|b, d, x_1) f(b, c, d) ds \\
&= \int E[Y|a, b, c, d, x_2] f(a|b, d, x_1) f(b, c, d) ds \\
&= \int (\alpha_1 a + \alpha_2 x_2 + \alpha_3 b + \alpha_4 c + \alpha_5 d) f(a|b, d, x_1) f(b, c, d) ds \\
&= \alpha_1 \int E[A|b, d, x_1] f(b, d) db dd + \alpha_2 x_2 + \int (\alpha_3 b + \alpha_4 c + \alpha_5 d) f(b, c, d) db dc dd \\
&= \alpha_1 \gamma_1 x_1 + \alpha_2 x_2 + (\alpha_1 \gamma_3 + \alpha_3) E[B] + \alpha_4 E[C] + (\alpha_1 \gamma_2 + \alpha_5) E[D].
\end{align*} \]
• Graphical necessary condition for identification of $f(y|do(x))$.
• Necessary and sufficient graphical criterion for identification of causal effects.
• Proposition on the “necessity” of adjustment.

Thanks!