Causal effects in MPDAGs: identification and efficient estimation

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Goal

- Estimate the total causal effect of A on Y

Observational data

Randomized control studies

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  $f(y|do(a)) \neq f(y|a)$.

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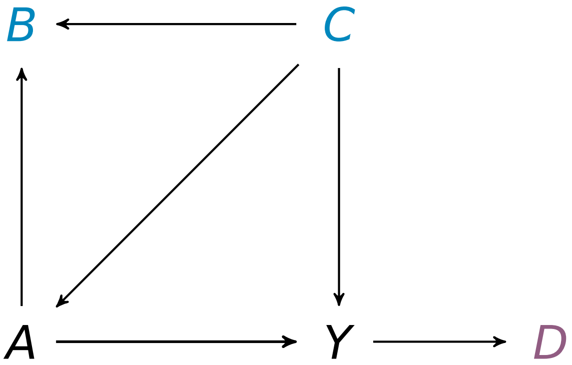
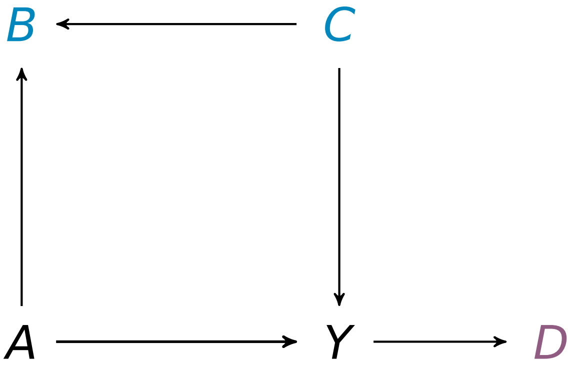
Observational Causal DAG



Causal Directed Acyclic Graph (DAG) $\mathcal{D}$.

Interventional Causal DAG



Causal DAG $\mathcal{D}$ **after a "do"-intervention on** A**.**

DAGs and distributions

Interventional density

- $do(a)$: an intervention that sets variables A to a.
- Observational density $f(v)$, Interventional density $f(v|do(a))$.

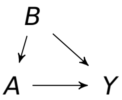
DAGs and distributions

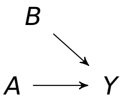
Interventional density

- $do(a)$: an intervention that sets variables A to a.
- Observational density $f(v)$, Interventional density $f(v|do(a))$.

- A DAG $\mathcal{D}$ is causal if for all observational and interventional densities:

$$f(v) = \prod_{V_j \in V} f(v_j|pa(v_j, \mathcal{D})) \quad \text{and} \quad f(v|do(a)) = \prod_{V_j \in V \setminus A} f(v_j|pa(v_j, \mathcal{D})).$$





$$f(b, a, y) = f(y|b, a)f(a|b)f(b)$$

$$f(b, y|do(a)) = f(y|b, a)f(b)$$

$$f(b, y|a) = f(y|b, a)f(b|a) \neq f(b, y|do(a))$$

How to define a causal effect?

Total causal effect

- Total causal effect - τ_{ay} - is *some functional* of $f(y|do(a))$, $P(Y|do(a))$.
- Examples: $E[Y|do(A = a + 1)] - E[Y|do(A = a)]$, $\frac{\partial}{\partial a}E(Y|do(a))$, OR, RR...

Identifiability

- A causal effect is identifiable from observational data if

$$f(y|do(a)) \text{ is computable from } f(v).$$

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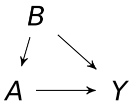
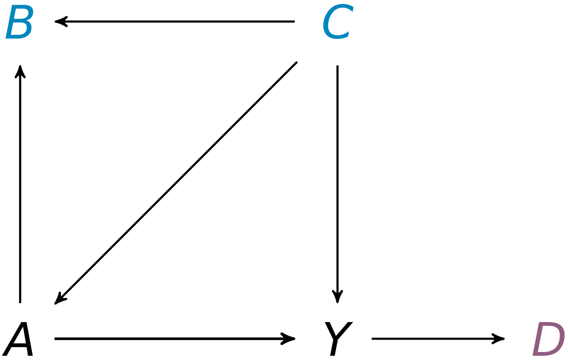
- Given the causal DAG, every total causal effect is identifiable.



$$f(y|do(a)) = \int f(b, y|do(a))db$$
$$= \int f(y|b, a)f(b)db.$$

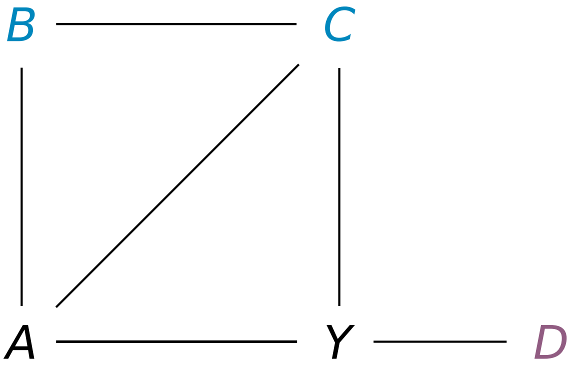
G-formula

Problem solved?



DAG $\mathcal{D}$.

Problem solved?



Completed Partially Directed Acyclic Graph (CPDAG) $\mathcal{C}$.

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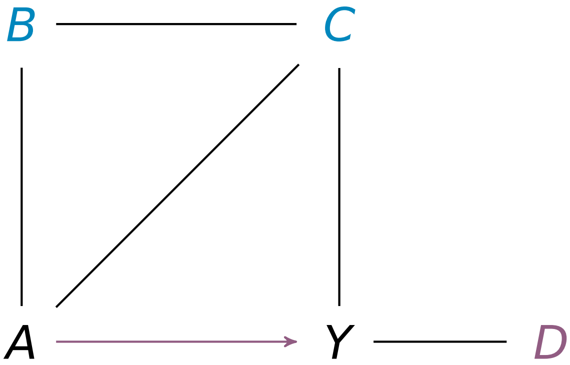

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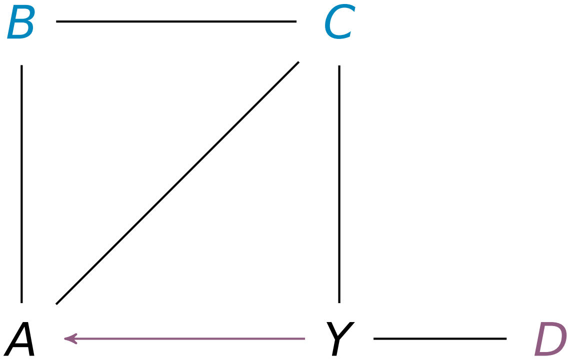

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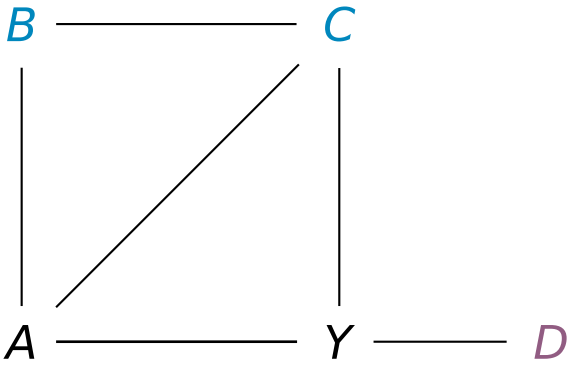

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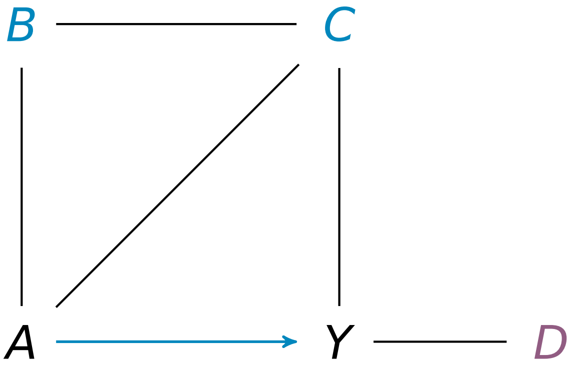
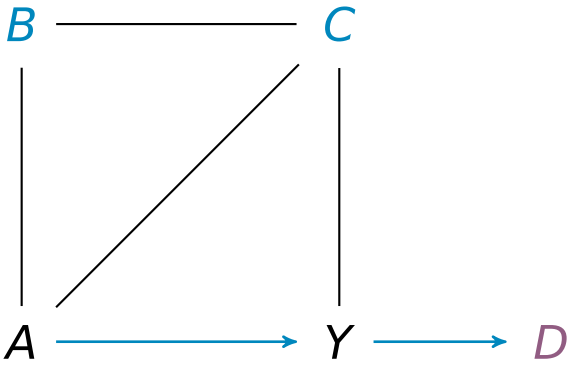


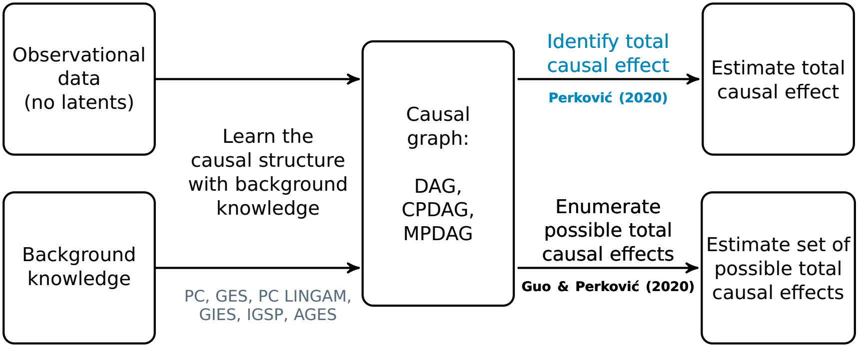
Partially Directed Acyclic Graph (PDAG).

Problem solved?



Maximally oriented Partially Directed Acyclic Graph (MPDAG) $\mathcal{G}$.

Framework

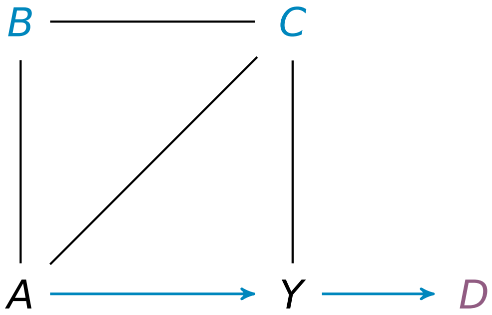


- PC (Spirtes et al, 1993), GES (Chickering, 2002)
- Adding background knowledge (Meek, 1995; TETRAD, Scheines et al., 1998), PC LINGAM (Hoyer et al., 2008), GIES (Hauser and Bühlmann, 2012), IGSP (Wang et al., 2017), etc.

Necessary and sufficient condition

Theorem (Perković, 2020)

The total causal effect of A on Y is identifiable in MPDAG $\mathcal{G}$ if and only if **all proper possibly causal paths** from A to Y start with a directed edge in $\mathcal{G}$.



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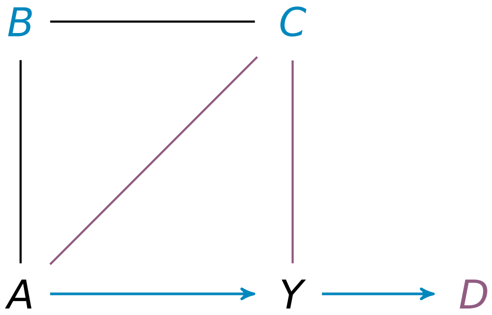

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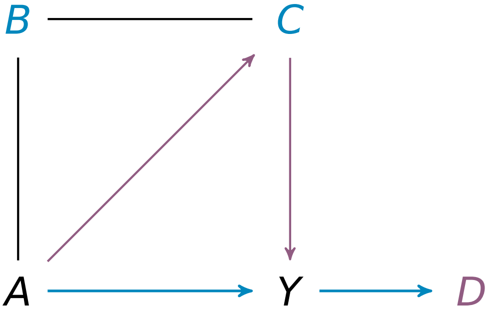

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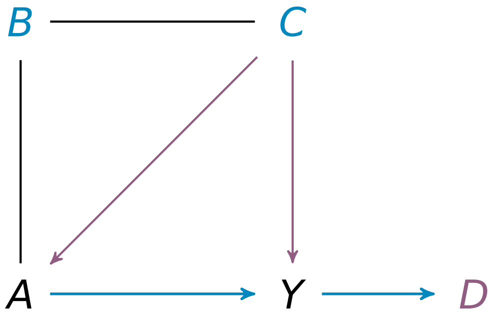

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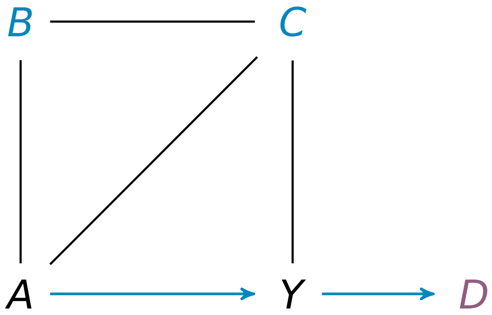

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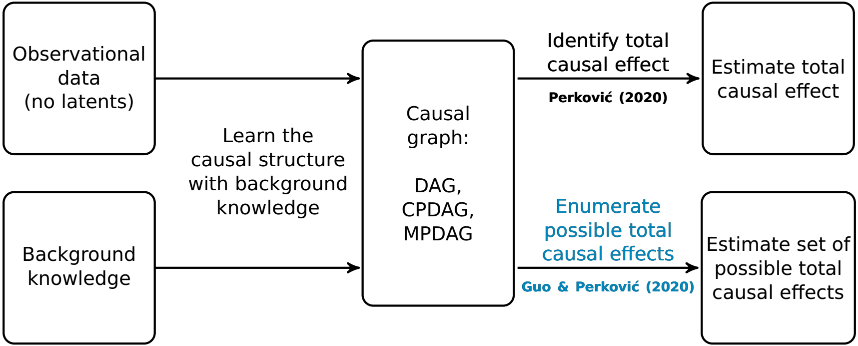

Observational data (no latents)
Background knowledge
Learn the causal structure with background knowledge
Causal graph:
DAG, CPDAG, MPDAG
Identify total causal effect
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Enumerate possible total causal effects
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Estimate total causal effect
Estimate set of possible total causal effects

Enumeration

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- Naive: enumerate all the DAGs in $[\mathcal{G}]$, and identify for each DAG.
  - Computationally prohibitive for large $|V|$ (the complete graph contains $|V|!$ DAGs); see also GIllispie et al (2002), Steinsky et al (2013).

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    - $|A| = 1$: IDA algorithm (Maathuis et al 2009).
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  - Yet, the total effect and $f(y|do(a))$ can be the same for two different parent sets!

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For an MPDAG $\mathcal{G}$, we look for sub-MPDAGs $\mathcal{G}_1, \ldots, \mathcal{G}_m$ such that

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1. **complete**: $[\mathcal{G}] = [\mathcal{G}_1] \dot\cup [\mathcal{G}_2] \dot\cup \cdots \dot\cup [\mathcal{G}_m]$
2. $f(y|\mathrm{do}(a))$ is identifiable under each $\mathcal{G}_i$
3. **minimal**: maps $f \mapsto f(y|do(a))$ are distinct under each $\mathcal{G}_i$ (identification formulae are distinct)
   $\Rightarrow$ possible causal effects $f \mapsto \frac{\partial}{\partial a} \mathbb{E}(Y|\mathrm{do}(A) = a)$ are distinct functionals!

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A natural algorithm is to recursively orient the undirected edges attached to A on **proper possibly causal paths** to Y.

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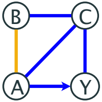
3. Recurse on $\mathcal{G}_1$ and $\mathcal{G}_2$ until $f(y|\text{do}(a))$ is identified

MPDAG($\mathcal{G}, R$) adds orientations R to $\mathcal{G}$ and completes Meek orientation rules.

Optimal enumeration

Orienting $A - B$ first ...



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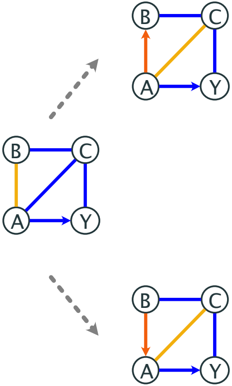

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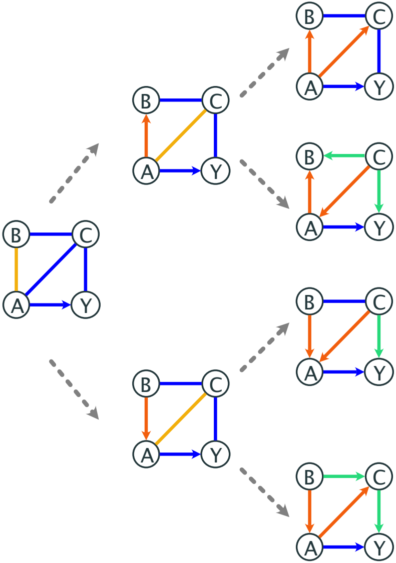

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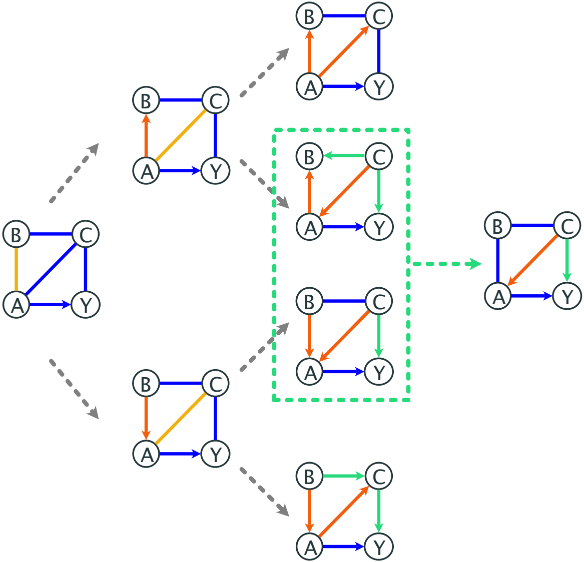

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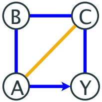
Orienting $A - C$ first ...



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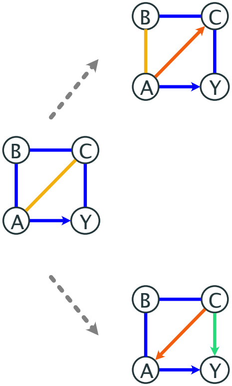

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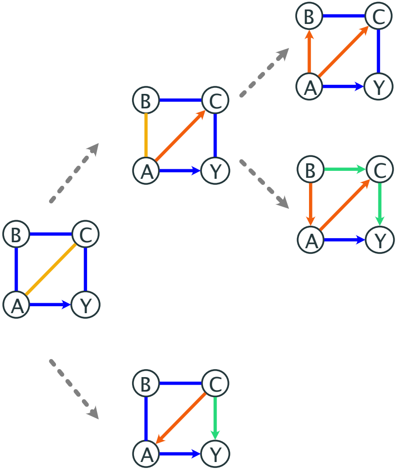
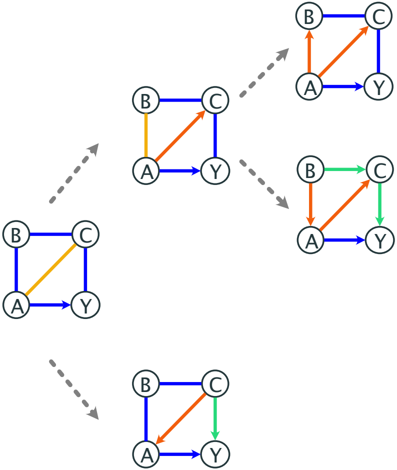


Optimal enumeration

Orienting $A - C$ first ...

- $A - C$ should be oriented first because the *status* of $A - B - C - Y$ depends on $A - C - Y$.



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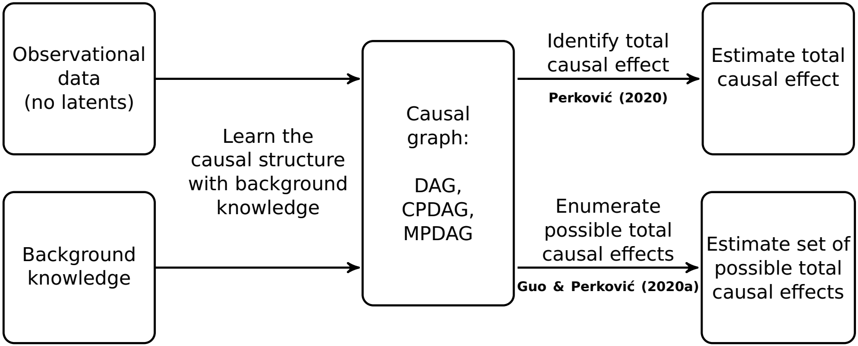
In contrast, the IDA algorithm will output 4 effects for this example, but two of them are different estimates of the same possible effect!

Framework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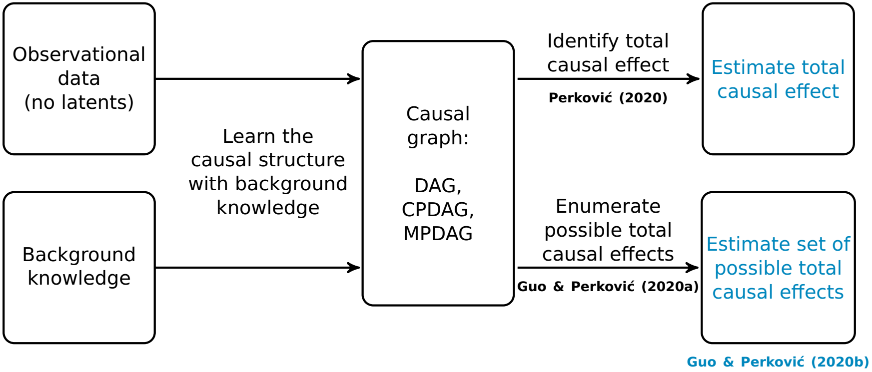
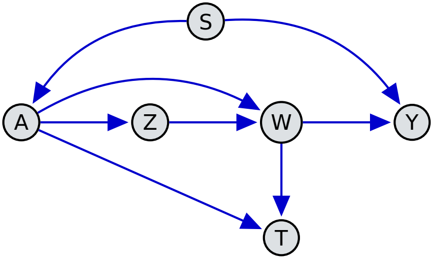


- In the following, we further assume **linearity** in the data generating mechanism.

Causal DAG, linear SEM



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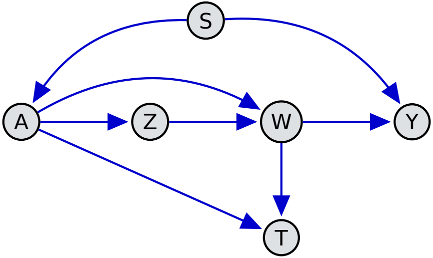

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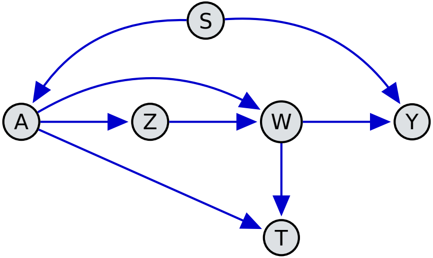

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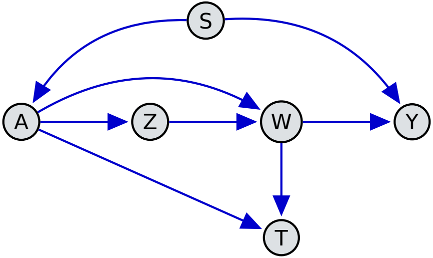
Causal DAG, linear SEM



- Suppose $\mathcal{D}$ is the underlying causal DAG.
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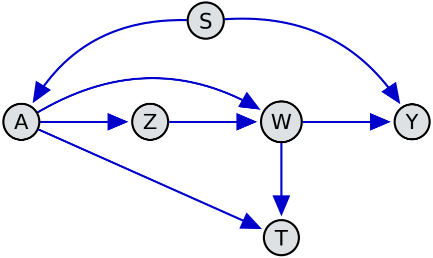

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- Under no unobserved confounder, the errors are **mutually independent**.

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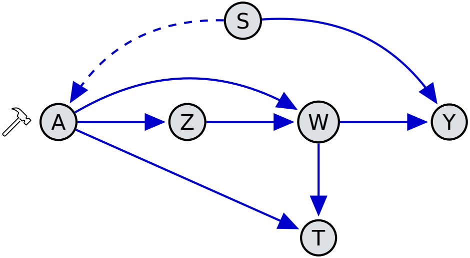

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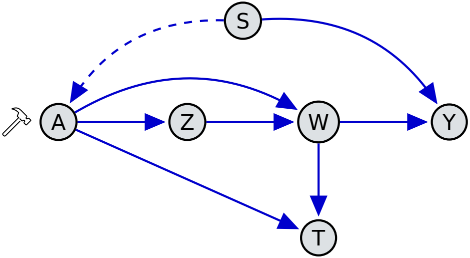


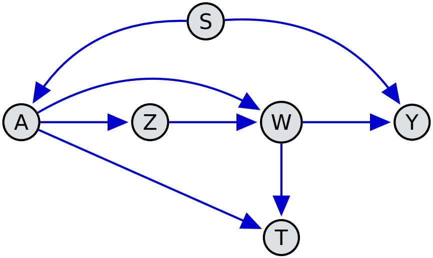
The total effect τ_{AY} is defined as the slope of $x_a \mapsto \mathbb{E}[X_Y | \text{do}(X_A = x_a)]$, given by a sum-product of Wright (1934):

$$\tau_{AY} = \frac{\mathrm{d}}{\mathrm{d}x_a} \mathbb{E}[X_Y | \text{do}(X_A = x_a)] = (\gamma_{AZ}\gamma_{ZW} + \gamma_{AW})\gamma_{WY}.$$

Estimation

Our task is to estimate τ_{AY} from n iid observational sample generated by a linear SEM associated with causal DAG $\mathcal{D}$, given that

$$\mathcal{D} \in [\mathcal{G}] \text{ for MPDAG } \mathcal{G}, \quad \tau_{AY} \text{ is identifiable from } \mathcal{G}.$$



DAG $\mathcal{D}$

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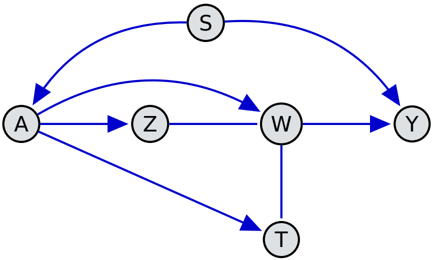
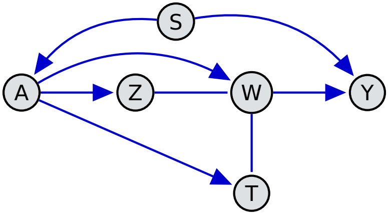


MPDAG $\mathcal{G}$

Buckets



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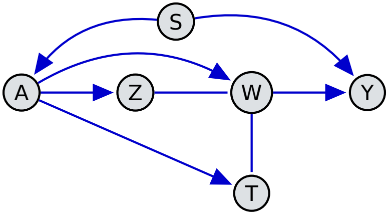

- Let “buckets” be the undirected connected components of MPDAG $\mathcal{G}$:

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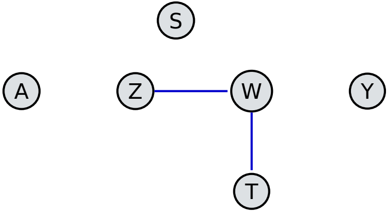

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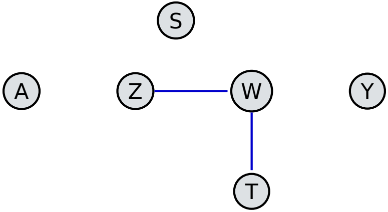

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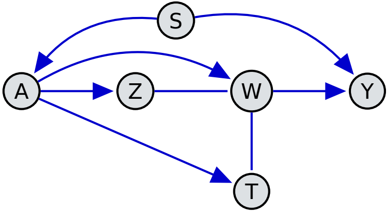

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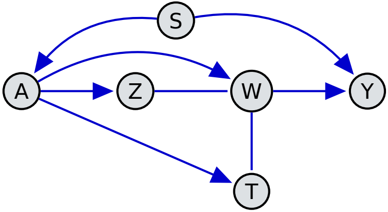

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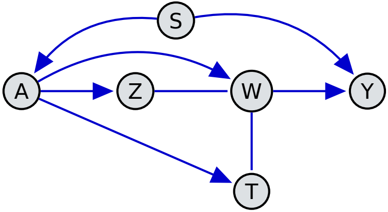

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- We use this to reparametrize the SEM.

Block-recursive reparametrization

Proposition (Block-recursive form, Guo and Perković, 2020)

Let $B_1, \ldots, B_K$ be the ordered bucket decomposition of V in MPDAG $\mathcal{G}$. Then

$$X = \Lambda^\intercal X + \varepsilon, \qquad \Lambda = (\lambda_{ij}), j \in B_k,\ i \notin \mathrm{pa}(B_k, \mathcal{G}) \quad \Rightarrow \quad \lambda_{ij} = 0,$$

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  - This is a special case of "seemingly unrelated regression" under the **restrictive property**.

Efficient estimator

If τ_{AY} is identifiable given MPDAG $\mathcal{G}$, the **$\mathcal{G}$-regression estimator** is defined as:

$$\hat{\tau}_{AY}^{\mathcal{G}} := \hat{\Lambda}_{A,S}^{\mathcal{G}} \left[(I - \hat{\Lambda}_{S,S}^{\mathcal{G}})^{-1} \right]_{S,Y},$$

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Theorem ($\mathcal{G}$-regression, Guo and Perković, 2020)

Then for any regular estimator $\hat{\tau}_{AY}$ that only uses the **first two moments** of the data, it holds that

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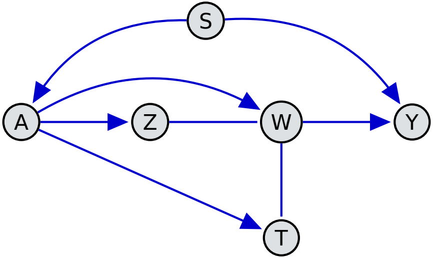
This includes estimators in the literature:

- covariate adjustment (Henckel et al., 2019, Witte et al., 2020),
- recursive regressions (Nandy et al., 2017, Gupta et al., 2020),
- modified Cholesky decomposition (Nandy et al., 2017).

Efficient estimator

The efficiency is achieved by exploiting the conditional independences encoded in $\mathcal{G}$.



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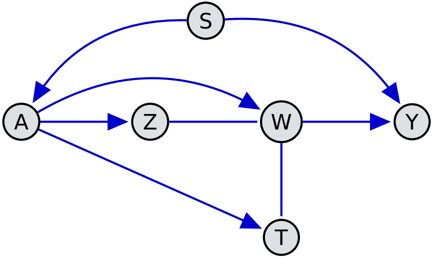

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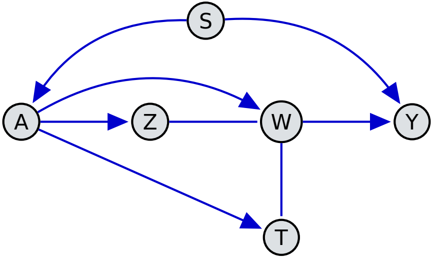

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adjustment estimator

$$\hat{\tau}_{AY}^{\text{adj}} = \hat{\beta}_{AY} \text{ from } Y \sim A + S.$$

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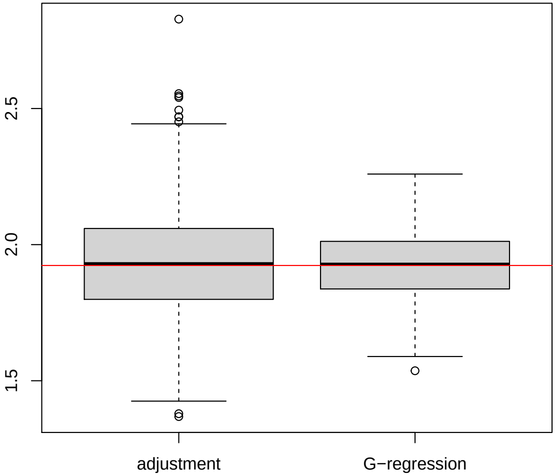


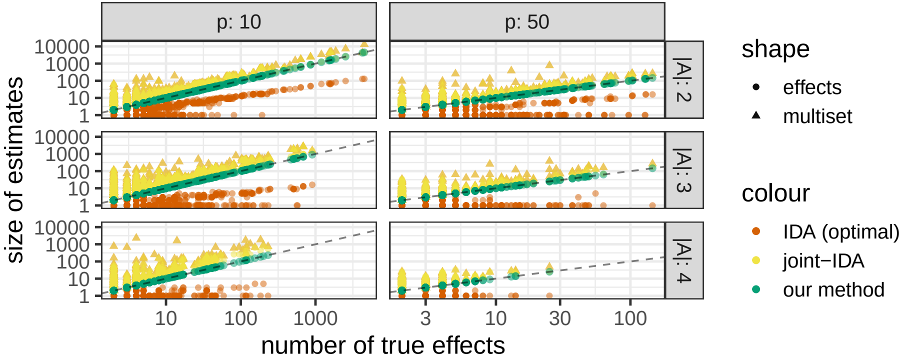
$n = 100$, t_5 errors.

Simulation: relative efficiency

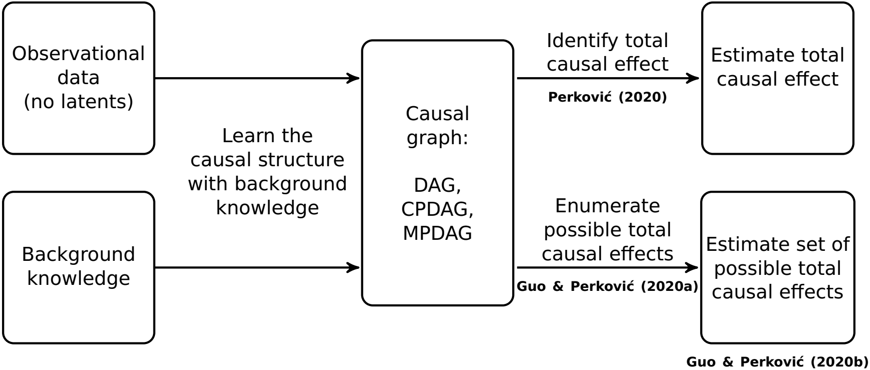
Table: Geometric average of squared errors relative to $\mathcal{G}$-regression:
`adj.O`: optimal adjustment
`IDA.M`: IDA (Cholesky)
`IDA.R`: IDA (recursive regression)

	$\|V\|=20$		$\|V\|=50$		$\|V\|=100$	
$\|A\|$	$n=100$	$n=1000$	$n=100$	$n=1000$	$n=100$	$n=1000$
`adj.O`						
1	1.3	1.3	1.4	1.3	1.5	1.5
2	3.4	4.2	4.7	4.9	4.2	4.5
3	6.3	5.9	7.4	7.2	7.8	8.0
4	9.3	9.3	12	14	12	12
`IDA.M`						
1	20	19	61	48	103	108
2	62	65	220	182	293	356
3	93	119	354	396	749	771
4	154	222	533	895	1188	1604
`IDA.R`						
1	20	19	61	48	103	108
2	33	38	121	113	176	199
3	30	39	171	135	342	312
4	48	50	187	214	405	432

Simulation: size of possible effects



Final remarks



- **R package** `eff`^{2}: github.com/richardkwo/eff2
- **Efficient estimation beyond linear SEMs?** We are working on it!

Thanks!