Total causal effects in MPDAGs: identification and minimal enumeration

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Goal

- Estimate the total causal effect of A on Y

Observational data

Randomized control studies

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  - the change in Y due to $do(a)$-
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- $do(a)$: an intervention that sets variables A to a.

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  $f(y|do(a)) \neq f(y|a)$.

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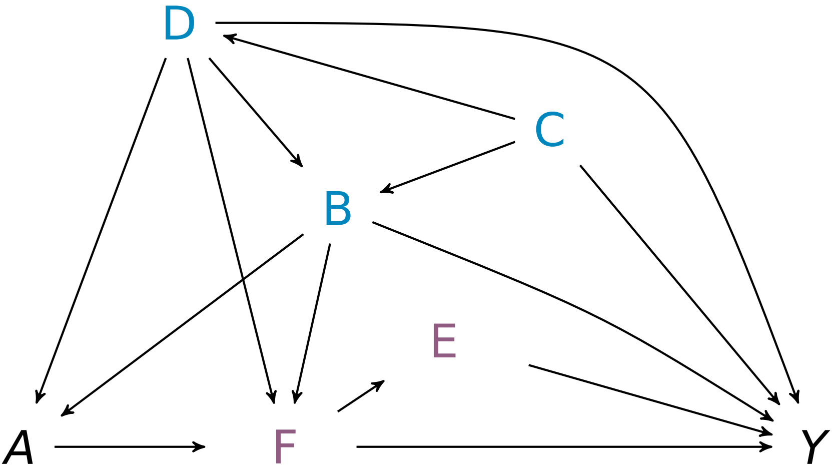
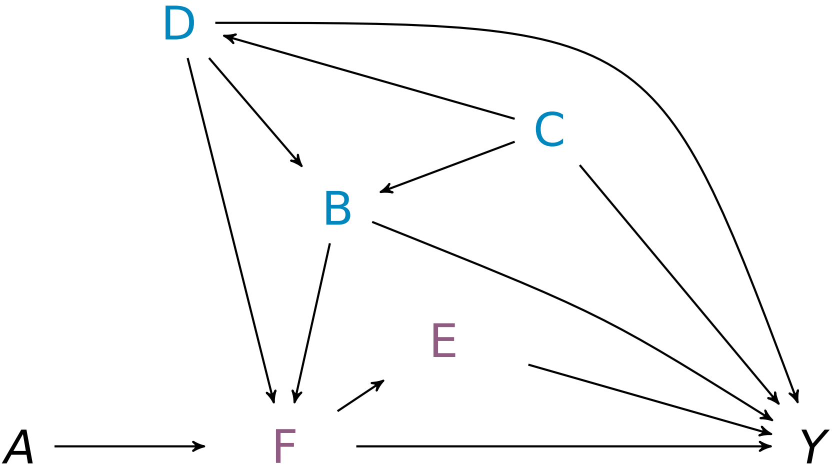
Observational Causal DAG



Causal Directed Acyclic Graph (DAG) $\mathcal{D}$.

Interventional Causal DAG



Causal DAG $\mathcal{D}$ **after a "do"-intervention on** A**.**

DAGs and distributions

Interventional density

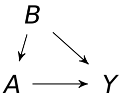
- $do(\mathbf{a})$: an intervention that sets variables $\mathbf{A}$ to $\mathbf{a}$.
- Observational density $f(\mathbf{v})$, Interventional density $f(\mathbf{v}|do(\mathbf{a}))$.

Interventional density

- $do(\mathbf{a})$: an intervention that sets variables **A** to **a**.
- Observational density $f(\mathbf{v})$, Interventional density $f(\mathbf{v}|do(\mathbf{a}))$.

- A DAG $\mathcal{D}$ is causal if for all observational and interventional densities:

$$f(\mathbf{v}) = \prod_{V_j \in \mathbf{V}} f(v_j|pa(v_j, \mathcal{D})) \quad \text{and} \quad f(\mathbf{v}|do(\mathbf{a})) = \prod_{V_j \in \mathbf{V}\setminus\mathbf{A}} f(v_j|pa(v_j, \mathcal{D})).$$



$$f(b, a, y) = f(y|b, a)f(a|b)f(b)$$

$$f(b, y|do(a)) = f(y|b, a)f(b)$$

$$f(b, y|a) = f(y|b, a)f(b|a) \neq f(b, y|do(a))$$

How to define a causal effect?

Total causal effect

- Total causal effect - $\tau_{\mathbf{ay}}$ - is *some functional* of $f(\mathbf{y}|do(\mathbf{a}))$, $P(\mathbf{Y}|do(\mathbf{a}))$.
- Examples: $E[Y|do(A = a + 1)] - E[Y|do(A = a)]$, $\frac{\partial}{\partial a}E(Y|do(a))$, OR, RR...

Identifiability

- A causal effect is identifiable from observational data if

$$f(\mathbf{y}|do(\mathbf{a})) \text{ is computable from } f(\mathbf{v}).$$

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- A causal effect is identifiable from observational data if

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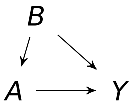
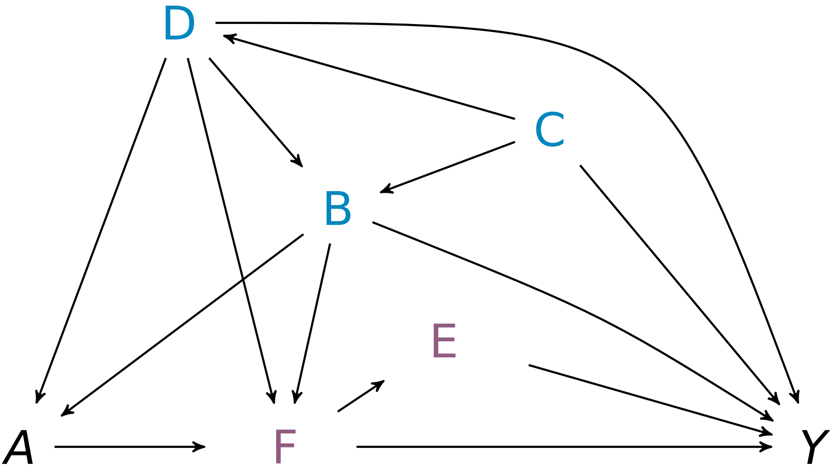
- Given the causal DAG, every total causal effect is identifiable.



$$f(y|do(a)) = \int f(b, y|do(a))db$$
$$= \int f(y|b, a)f(b)db.$$

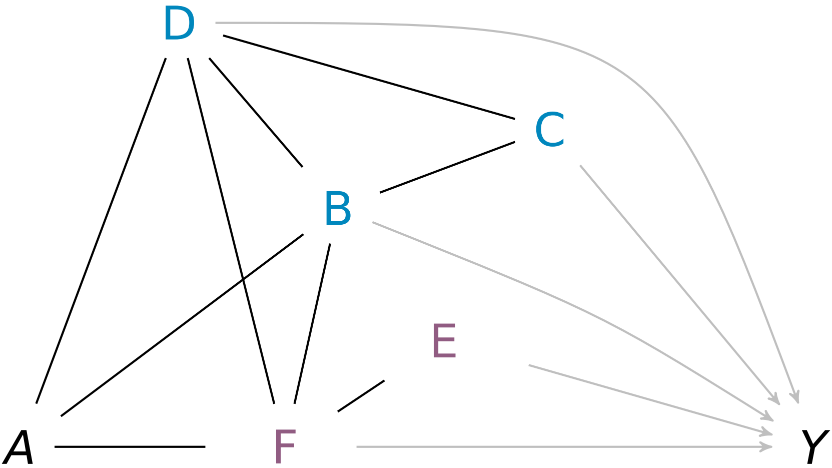
(Generalized) G-formula

Problem solved?



DAG $\mathcal{D}$.

Problem solved?



Completed Partially Directed Acyclic Graph (CPDAG) $\mathcal{C}$.

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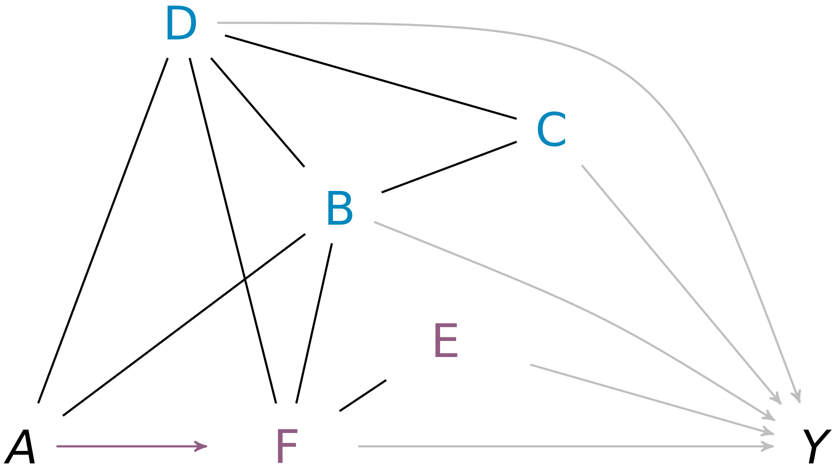

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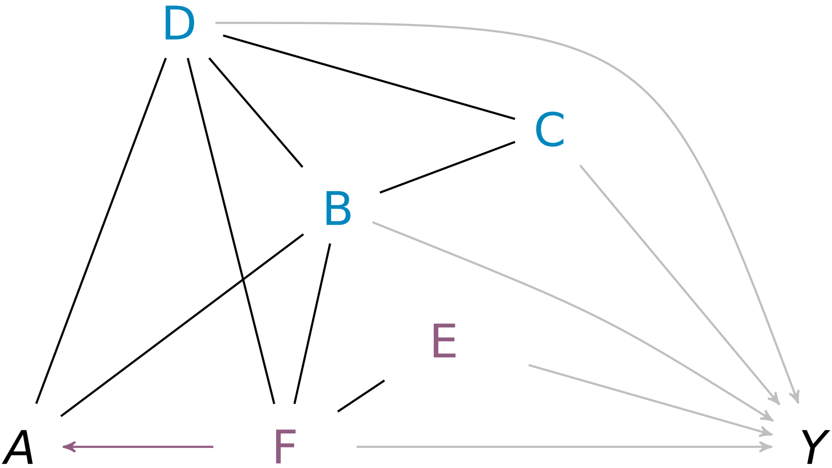

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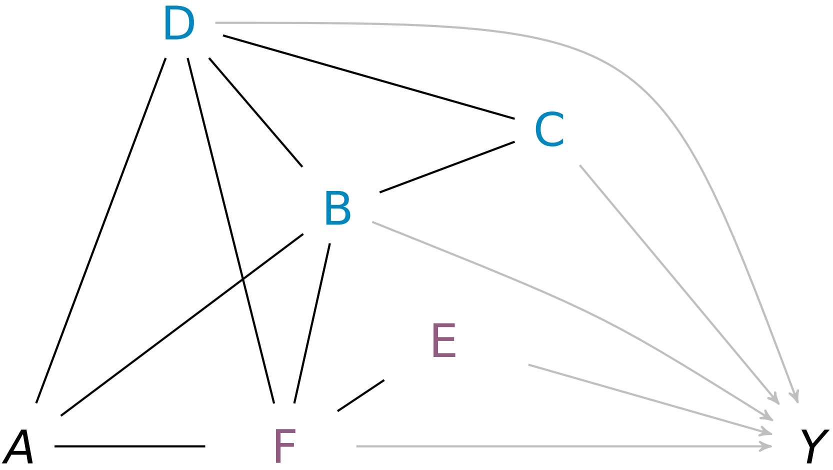

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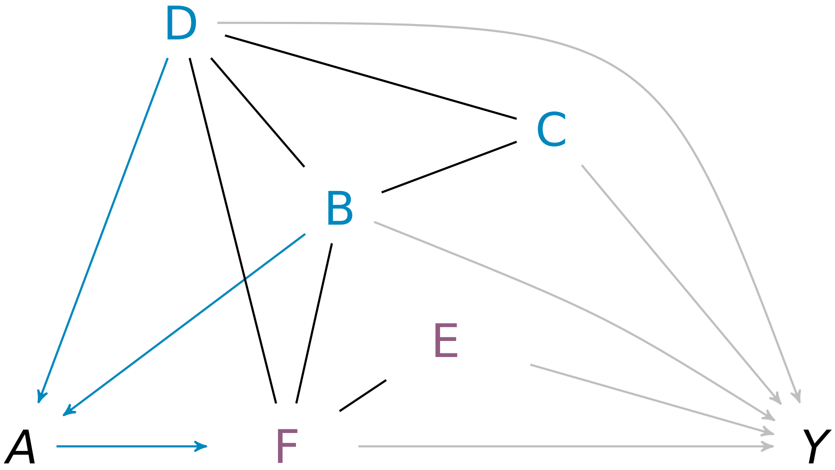
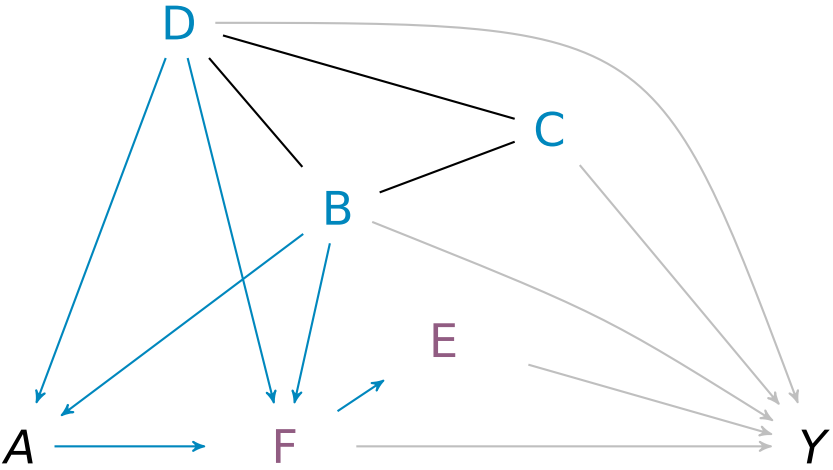


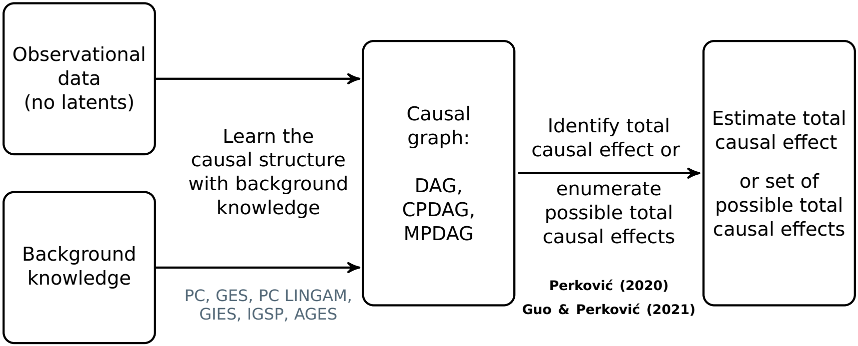
PDAG with Bg. Knowledge.

Problem solved?



Maximally oriented Partially Directed Acyclic Graph (MPDAG) $\mathcal{G}$.

Framework



- PC (Spirtes et al, 1993), GES (Chickering, 2002) + Adding background knowledge (Meek, 1995; TETRAD, Scheines et al., 1998), PC LINGAM (Hoyer et al., 2008), GIES (Hauser and Bühlmann, 2012), IGSP (Wang et al., 2017), etc.
- Other framing: start with a DAG and remove some directional information while keeping the orientations closed under Meek orientation rules (Meek, 1995).

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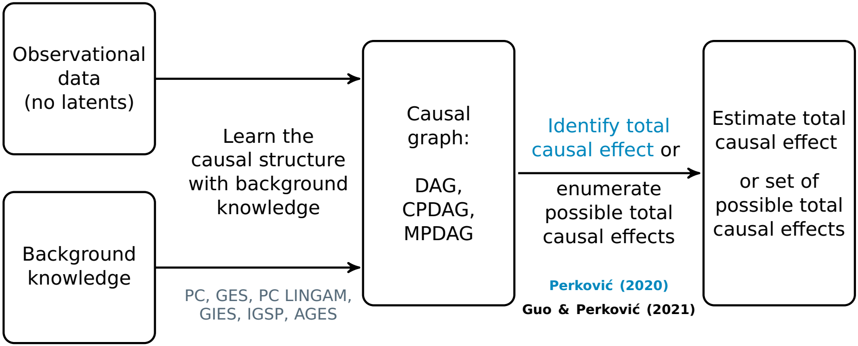

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Overview of graphical criteria for identification

Graphical criterion	DAG	CPDAG	MPDAG
Adjustment (Pearl '93, Shpitser et al '10, Perković et al '15, '17, '18)	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
G-formula, Truncated Factorization (Robins '86, Pearl '93, Spirtes '93)	$\Leftrightarrow$		

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Adjustment: Z is an adjustment set if

$$f(\mathbf{y}|do(\mathbf{a})) = \int f(\mathbf{y}|\mathbf{a},\mathbf{z})f(\mathbf{z})d\mathbf{z}$$

Truncated Factorization: Let $\mathbf{V}' = \mathbf{V} \setminus \{\mathbf{A} \cup \mathbf{Y}\}$, then

$$f(\mathbf{y}|do(\mathbf{a})) = \int \prod_{V_i \in \mathbf{V}\setminus\mathbf{A}} f(v_i|pa(v_i,\mathcal{D}))d\mathbf{v}'.$$

Does an adjustment set always exist?

If $\mathbf{A} = \{A\}$, $\mathbf{Y} = \{Y\}$:

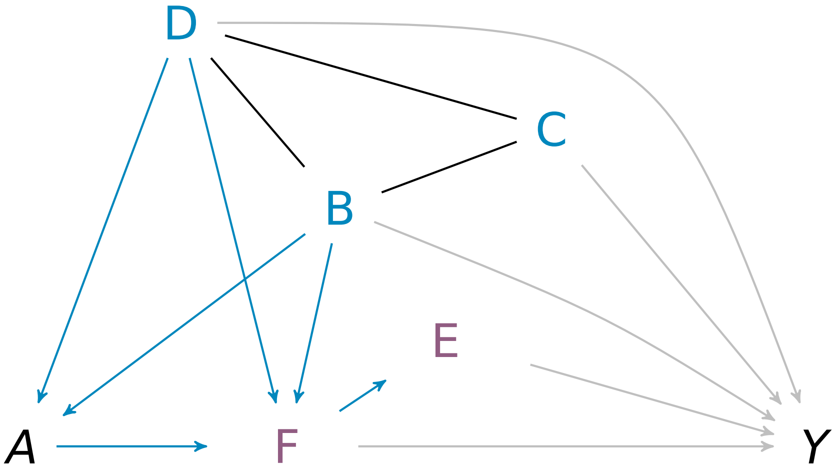
Proposition (Perković, 2020)

If $Y \notin Pa(A, \mathcal{G})$, then an adjustment set relative to (A, Y) exists in the MPDAG $\mathcal{G}$, if and only if the $f(y|do(a))$ is identifiable given $\mathcal{G}$.

What about for $|\mathbf{A}| > 1$, or $|\mathbf{Y}| > 1$?
Does an adjustment set always exist?

No. Not even in a DAG.

Joint Effects



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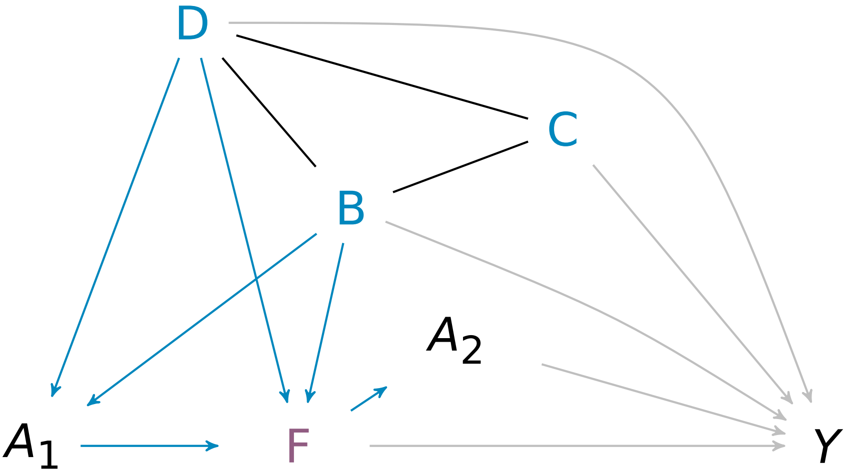

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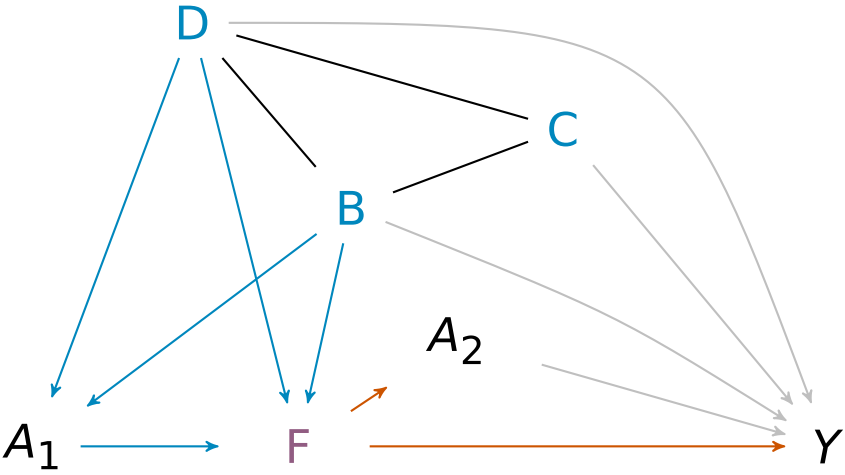

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Causal identification formula (Perković '20)	$\Leftrightarrow$	$\Leftrightarrow$	$\Leftrightarrow$

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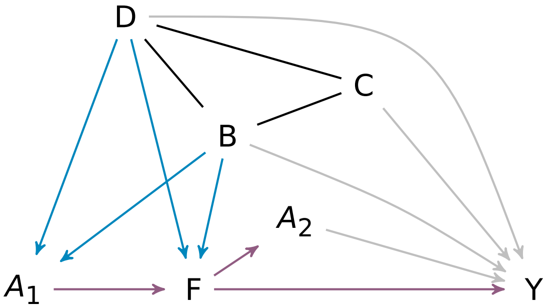
Necessary and sufficient condition

Theorem (Perković, 2020)

The total causal effect of **A** on **Y** is identifiable in MPDAG $\mathcal{G}$ if and only if **all proper possibly causal paths** from **A** to **Y** start with a directed edge in $\mathcal{G}$.



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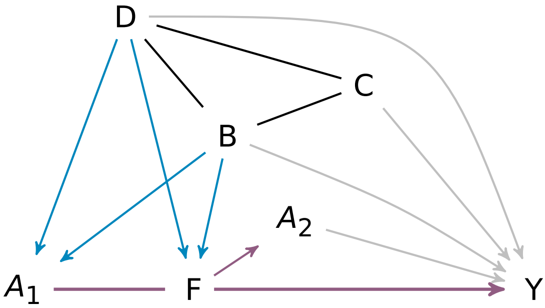

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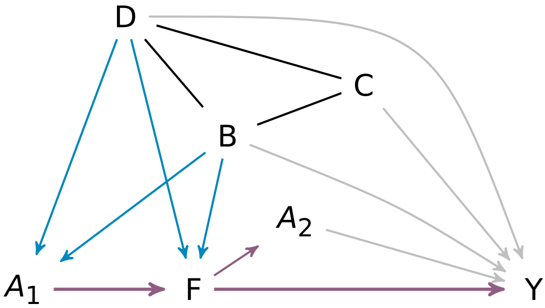

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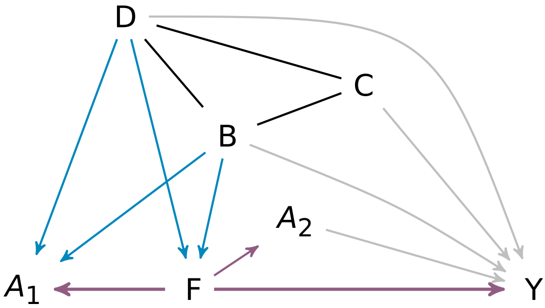


Causal identification formula

Theorem (Perković, 2020)

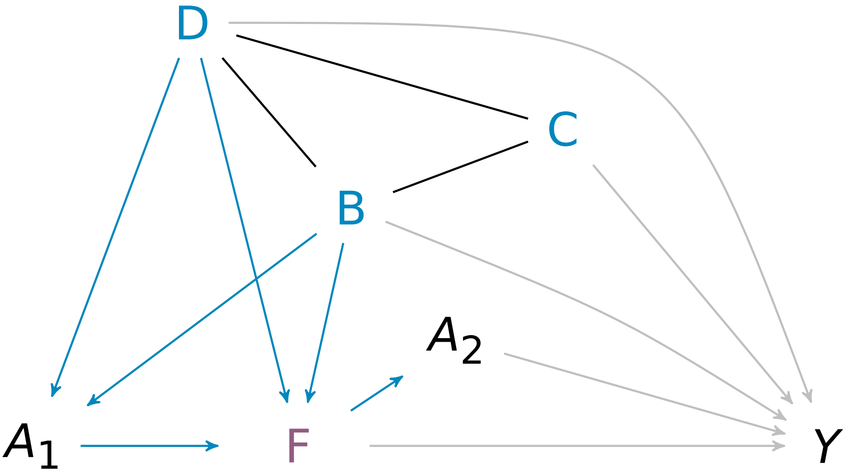
If **all proper possibly causal paths** from $\mathbf{A}$ to $\mathbf{Y}$ start with a directed edge in $\mathcal{G}$, then

$$f(\mathbf{y}|do(\mathbf{a})) = \int \prod_{i=1}^{k} f(\mathbf{s_i}|pa(\mathbf{s_i}, \mathcal{G}))d\mathbf{s},$$

where $\mathbf{S} = an(\mathbf{Y}, \mathcal{G}_{\mathbf{V}\setminus\mathbf{A}}) \setminus \mathbf{Y}$,
and $(\mathbf{S_1}, \dots, \mathbf{S_k})$ is a partition of $\mathbf{S} \cup \mathbf{Y}$ into undirected connected sets in $\mathcal{G}$.

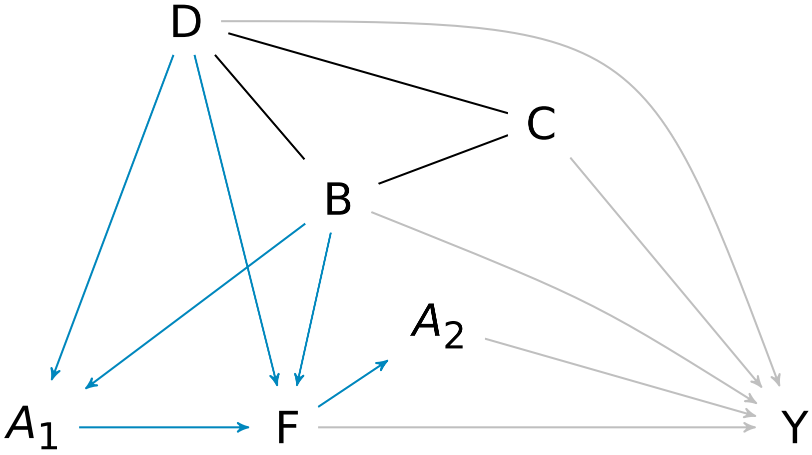
- $\mathbf{S} \cup \mathbf{Y} = an(\mathbf{Y}, \mathcal{G}_{\mathbf{V}\setminus\mathbf{A}})$ - nodes that have a causal path to $\mathbf{Y}$ that is not through $\mathbf{A}$.
- $(\mathbf{S_1}, \dots, \mathbf{S_k})$ - maximal connected components of $\mathbf{S} \cup \mathbf{Y}$ in the induced undirected subgraph of $\mathcal{G}$.

How to use the causal identification formula?



$$f(y|do(a_1, a_2)) = \int f(y|f, b, c, d, a_2) f(f|b, d, a_1) f(b, c, d) \, df \, db \, dc \, dd$$

$f(y|do(a_1, a_2)) = ?$



- $\mathbf{S} = an(Y, \mathcal{G}_{\mathbf{V} \setminus \mathbf{A}}) \setminus \{Y\} =$

Partition of $\mathbf{S} \cup \{Y\} =$

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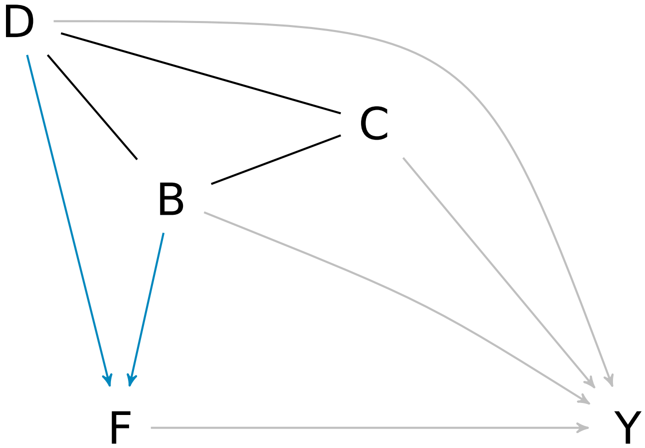

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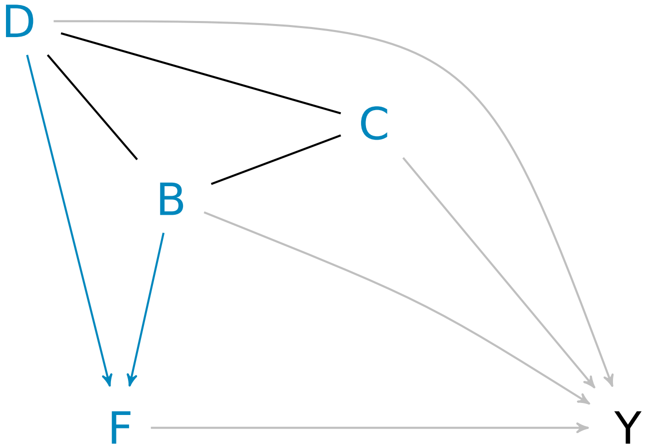

- $\mathbf{S} = an(Y, \mathcal{G}_{\mathbf{V}\setminus\mathbf{A}}) \setminus \{Y\} = \{F, B, C, D\}$, Partition of $\mathbf{S} \cup \{Y\} =$

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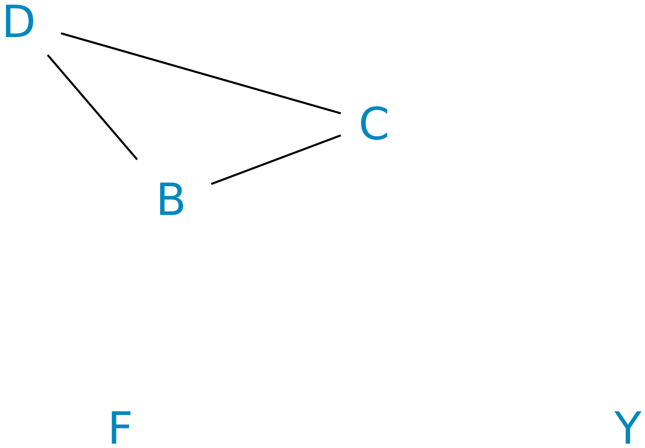

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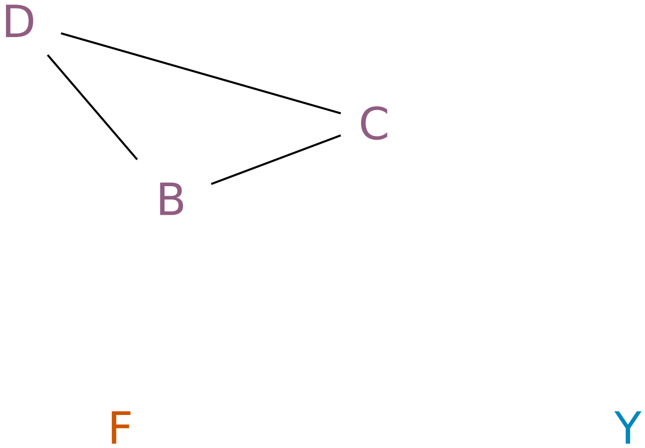

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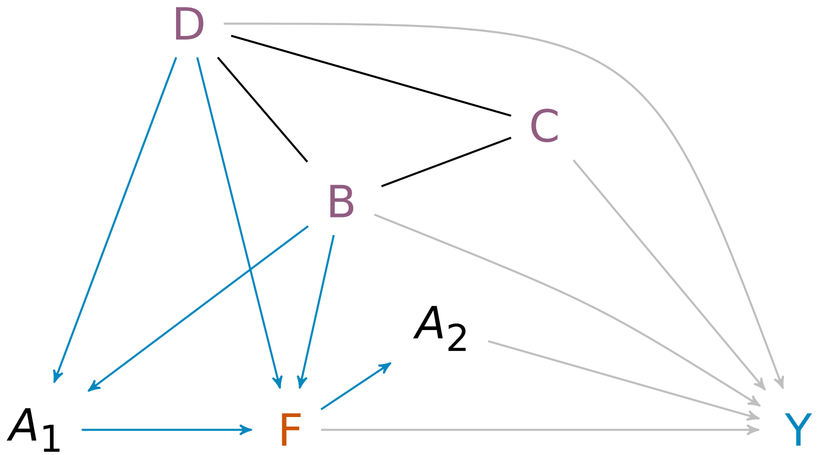

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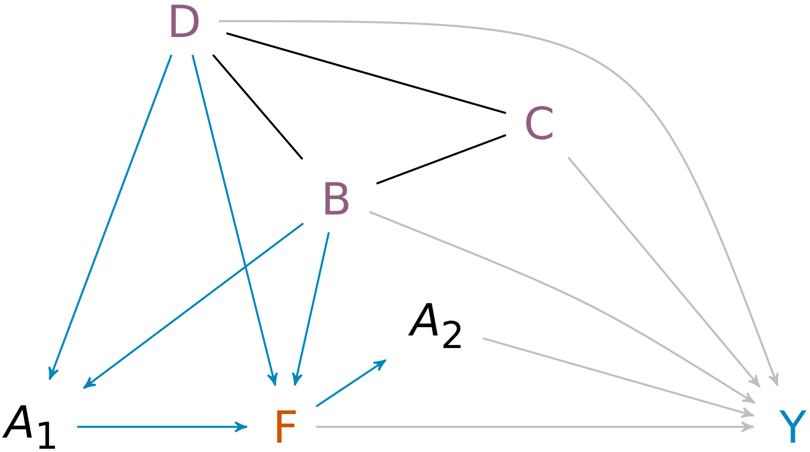

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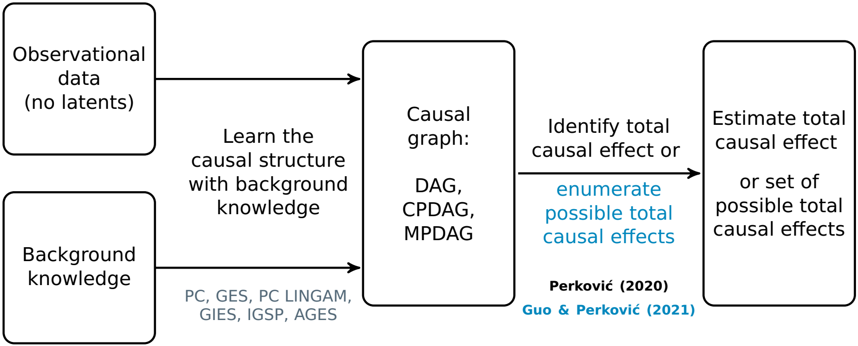
$$f(y|do(a_1, a_2)) = \int f(y, b, c, d, f, |do(a_1, a_2))d\mathbf{s} = \int f(y|b, c, d, f, a_2) f(f|b, a_1) f(b, c, d) d\mathbf{s}.$$

Framework



- What if the causal effect is not identifiable?

Theorem (Perković, 2020)

The total causal effect of **A** on **Y** is identifiable in MPDAG $\mathcal{G}$ if and only if **all proper possibly causal paths** from **A** to **Y** start with a directed edge in $\mathcal{G}$.

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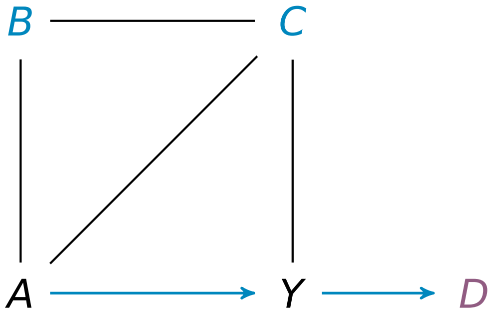

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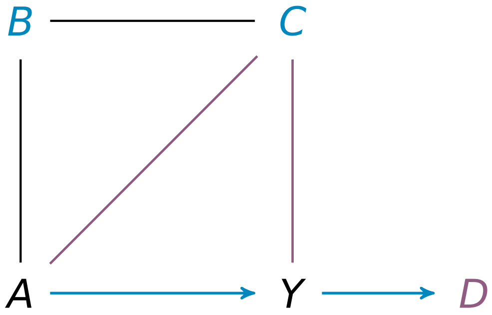

- How to enumerate all possible total causal effects?

IDA Enumeration Types

Theorem (Perković, 2020)

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For an MPDAG $\mathcal{G}$, we look for sub-MPDAGs $\mathcal{G}_1, \dots, \mathcal{G}_m$ such that

1. **complete**: $[\mathcal{G}] = [\mathcal{G}_1] \dot\cup [\mathcal{G}_2] \dot\cup \cdots \dot\cup [\mathcal{G}_m]$
2. $f(\mathbf{y}|\text{do}(\mathbf{a}))$ is identifiable under each $\mathcal{G}_i$

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   - all DAGs (Maathuis et al, ’09)
   - the valid parent sets of A (Maathuis et al, ’09, Nandy et al, ’17, Perković et al, ’17, Witte et al, ’20, Fang and He, ’20)

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   - orientation of $A-$ on possibly causal paths to Y (Liu et al, '20)

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   We could enumerate over

   - all DAGs (Maathuis et al, '09)
   - the valid parent sets of A (Maathuis et al, '09, Nandy et al, '17, Perković et al, '17, Witte et al, '20, Fang and He, '20)
   - orientation of $A-$ on possibly causal paths to Y (Liu et al, '20)
3. **minimal**: maps $f \mapsto f(\mathbf{y}|do(\mathbf{a}))$ are distinct under each $\mathcal{G}_i \Rightarrow$ possible causal effects $f \mapsto \frac{\partial}{\partial a_i} \mathbb{E}(\mathbf{Y}|\text{do}(\mathbf{A}) = \mathbf{a})$ are distinct functionals!

   - None of the above approaches are minimal!

Optimal enumeration

Theorem (Perković, 2020)

The total causal effect of $\mathbf{A}$ on $\mathbf{Y}$ is identifiable in MPDAG $\mathcal{G}$ if and only if **all proper possibly causal paths** from $\mathbf{A}$ to $\mathbf{Y}$ start with a directed edge in $\mathcal{G}$.

Input: MPDAG $\mathcal{G}$, $\mathbf{Y} \subset \mathbf{V}$ and $\mathbf{A} \subset \mathbf{V} \setminus \mathbf{Y}$.

Algorithm FirstTry

Optimal enumeration

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Algorithm FirstTry

1. Pick $A_1 - V_1$ such that $A_1 \in \mathbf{A}$ and there is a proper possibly causal path $A_1, V_1, \dots, Y_1$, $Y_1 \in \mathbf{Y}$.

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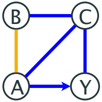
3. Recurse on $\mathcal{G}_1$ and $\mathcal{G}_2$ until $f(\mathbf{y}|\text{do}(\mathbf{a}))$ is identified

MPDAG$(\mathcal{G}, R)$ adds orientations R to $\mathcal{G}$ and completes Meek orientation rules.

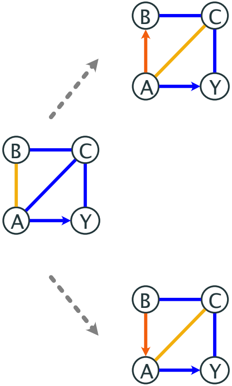
Optimal enumeration

Orienting $A - B$ first ...



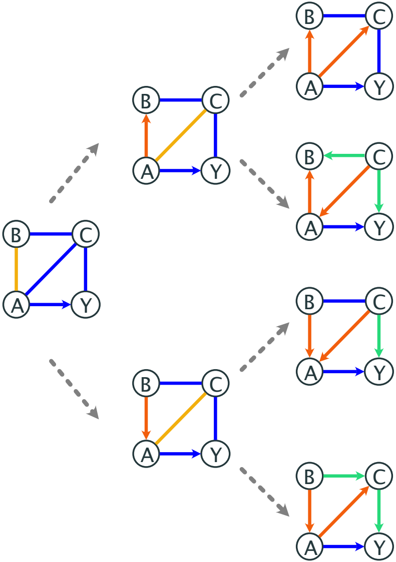
Optimal enumeration

Orienting $A - B$ first ...



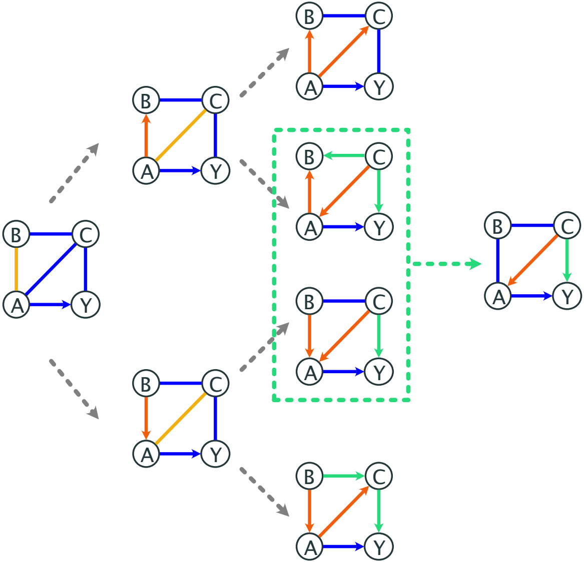
Optimal enumeration

Orienting $A - B$ first ...



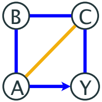
Optimal enumeration

Orienting $A - B$ first ...



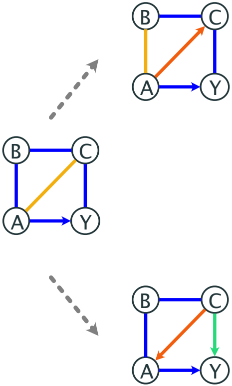
Optimal enumeration

Orienting $A - C$ first ...



Optimal enumeration

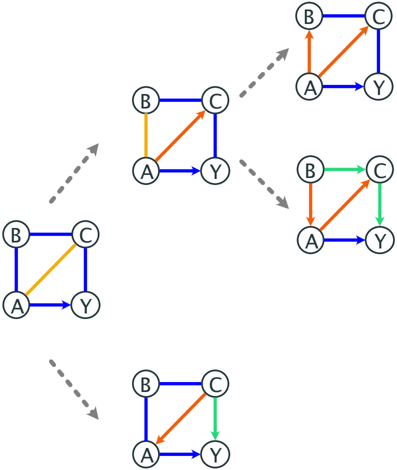
Orienting $A - C$ first ...



Optimal enumeration

Orienting $A - C$ first ...

- $A - C$ should be oriented first because the *status* of $A - B - C - Y$ depends on $A - C - Y$.



Optimal enumeration

Algorithm IDGraphs, (Guo & Perković, 2021)

1. Pick $A_1 - V_1$ such that $A_1 \in \mathbf{A}$ and $A_1, V_1, \dots, Y_1$, $Y_1 \in \mathbf{Y}$ is a shortest proper possibly causal path from $\mathbf{A}$ to $\mathbf{Y}$.
2. $\mathcal{G}_1 \leftarrow \text{MPDAG}(\mathcal{G}, A_1 \rightarrow V_1)$, $\mathcal{G}_2 \leftarrow \text{MPDAG}(\mathcal{G}, A_1 \leftarrow V_1)$
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Optimal enumeration

Algorithm IDGraphs, (Guo & Perković, 2021)

1. Pick $A_1 - V_1$ such that $A_1 \in \mathbf{A}$ and $A_1, V_1, \ldots, Y_1$, $Y_1 \in \mathbf{Y}$ is a shortest proper possibly causal path from **A** to **Y**.
2. $\mathcal{G}_1 \leftarrow$ MPDAG($\mathcal{G}, A_1 \rightarrow V_1$), $\mathcal{G}_2 \leftarrow$ MPDAG($\mathcal{G}, A_1 \leftarrow V_1$)
3. Recurse on $\mathcal{G}_1$ and $\mathcal{G}_2$ until identified

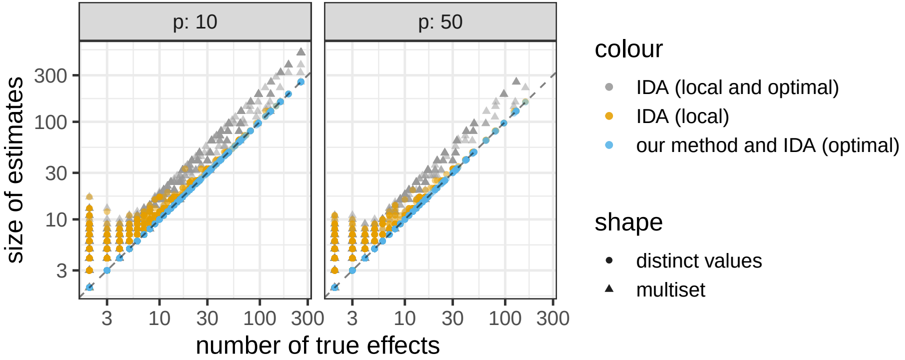
Theorem (Guo & Perković, 2021)

$(\mathcal{G}_1, \ldots, \mathcal{G}_m)$ output by the algorithm is **complete** and **minimal**.

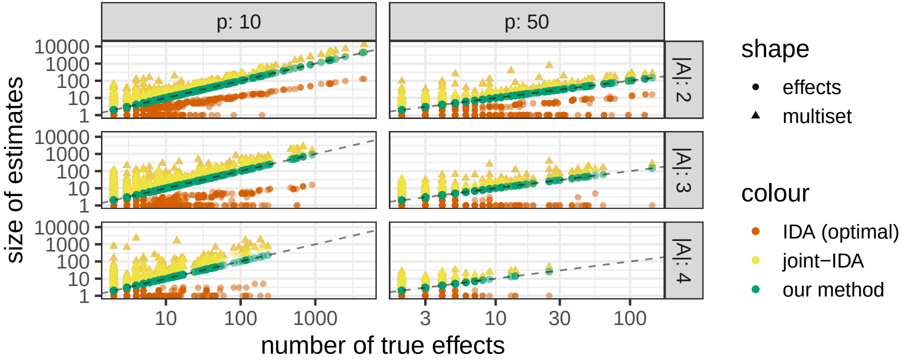
Hence, each $\mathcal{G}_i$ represents the minimal set of additional orientations required for a particular interventional distribution/possible effect!

In contrast, the existing algorithms will output 4 effects for this example, but two of them are different estimates of the same possible effect!

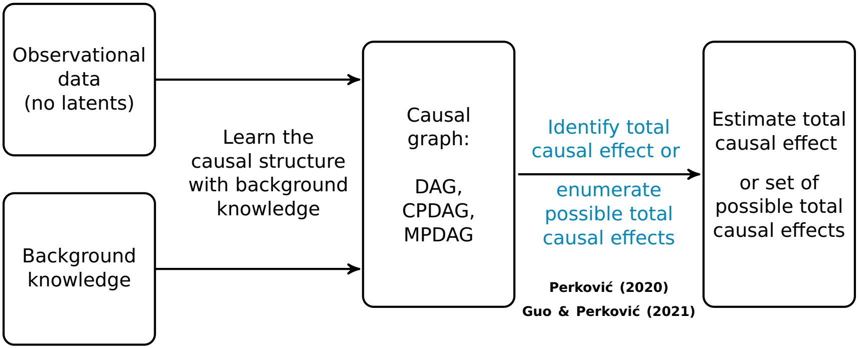
Simulation: size of possible effects



Simulation: size of possible effects



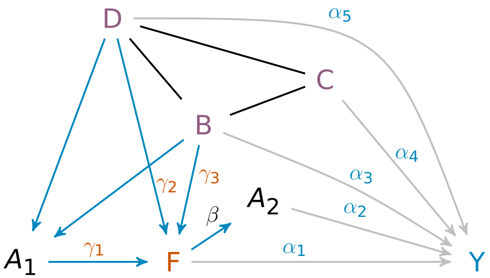
Framework



- **R package** `eff`^{2}: github.com/richardkwo/eff2

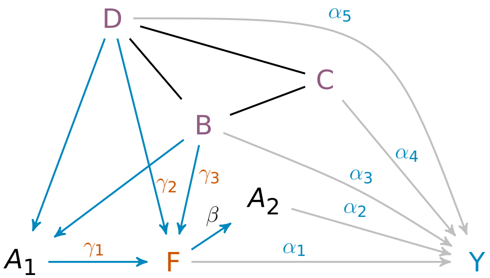
Thanks!

Estimation in the linear case



- $\tau_{y\mathbf{a}} = (\tau_{ya_1.a_2}, \tau_{ya_2.a_1})^T = (\alpha_1\gamma_1, \alpha_2)^T = (\frac{\partial E[Y|do(a_1,a_2)]}{\partial a_1}, \frac{\partial E[Y|do(a_1,a_2)]}{\partial a_2})^T$

Estimation in the linear case



- $\tau_{y\mathbf{a}} = (\tau_{ya_1.a_2}, \tau_{ya_2.a_1})^T = (\alpha_1\gamma_1, \alpha_2)^T = \left(\frac{\partial E[Y|do(a_1,a_2)]}{\partial a_1}, \frac{\partial E[Y|do(a_1,a_2)]}{\partial a_2}\right)^T$

$$E[Y|do(a_1,a_2))] = \int yf(y|b,c,d,f,a_2)f(f|b,d,a_1)f(b,c,d)d\mathbf{s}$$

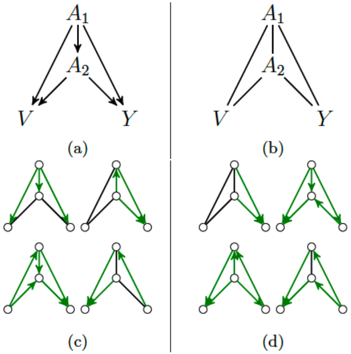
$$= \int E[Y|b,c,d,f,a_2]f(f|b,d,a_1)f(b,c,d)d\mathbf{s}$$

$$= \int (\alpha_1 f + \alpha_2 a_2 + \alpha_3 b + \alpha_4 c + \alpha_5 d)f(f|b,d,a_1)f(b,c,d)d\mathbf{s}$$

$$= \alpha_1 \int E[F|b,d,a_1]f(b,d)db\,dd + \alpha_2 a_2 + \int (\alpha_3 b + \alpha_4 c + \alpha_5 d)f(b,c,d)db\,dc\,dd$$

$$= \alpha_1\gamma_1 a_1 + \alpha_2 a_2 + (\alpha_1\gamma_3 + \alpha_3)E[B] + \alpha_4 E[C] + (\alpha_1\gamma_2 + \alpha_5)E[D].$$

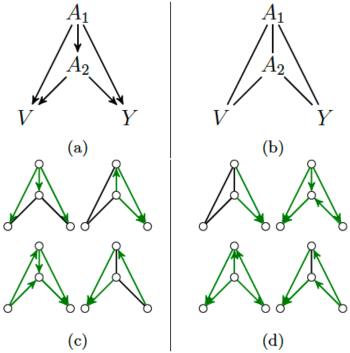
Simulation results



	A_1 on Y (c)
true effect	3
true poss. effects	$\{3, 2, 1.8, 0\}$
our method	$\{2.9, 2.1, 1.9, 0\}$
IDA (optimal)	$\{2.9, (2.1)^2, 1.9, 0\}$
IDA (local, collapsible)	$\{2.9, 2.1, 2.2, 1.9, 0\}$
joint-IDA	—

- Generated with a linear structural causal model with Gaussian errors and $n = 100$.
- $(a)^b$ denotes that a appears with multiplicity b.

Simulation results



	A_1, A_2 on Y (d)
true effect	(2,1)
true poss. effects	$\{(2,1), (3,0), (0,2), (0,0)\}$
our method	$\{(2.1, 0.9), (2.9, 0), (0, 1.9), (0, 0)\}$
IDA (optimal)	$\{(2.1, 0.9)^6, (0, 0)^2, (\texttt{NA}, \texttt{NA})^2\}$
IDA (local, collapsible)	—
joint-IDA	$\{(2.1, 0.9)^2, (2.2, 0.9), (1.9, 1.1),$ $(2.2, 1.1)^2, (0, 1.9), (2.9, 0), (0, 0)^2\}$

- Generated with a linear structural causal model with Gaussian errors and $n = 100$.
- $(a)^b$ denotes that a appears with multiplicity b.

Overview

	Comp. Cost	$\|A\|=1$	$\|A\|>1$	Duplicates
Naive - Enumerate all DAGs:				
global IDA (Maathuis et al, 2009)	$\mathcal{O}(\|V\|!)$	✓	-	Yes
global joint IDA (Nandy et al, 2017)	$\mathcal{O}(\|V\|!)$	✓	✓	Yes
Enumerate valid parent sets of A:				
local IDA (Maathuis et al, 2009, Fang & He, 2020)	$\mathcal{O}(2^{l(\mathcal{G})})$	✓	-	Yes
semi-local IDA, joint IDA (P. et al, 2017,Nandy et al, 2017)	$\mathcal{O}(2^{l(\mathcal{G})} poly(\|V\|))$	✓	✓	Yes
optimal IDA (Witte et al, 2020)	$\mathcal{O}(2^{l(\mathcal{G})} poly(\|V\|))$	✓	$\sim$	No
Enum. $A-$ on poss. causal paths to Y:				
collapsible IDA (Liu et. al, 2020)	$\mathcal{O}((\|V\|+\|E\|)2^{r(\mathcal{G})})$	✓	-	Yes

- $l(\mathcal{G})$ - # of undirected edges connected to A
- $r(\mathcal{G})$ - # of edges $A-$ on possibly causal paths to Y, $r(\mathcal{G}) \leq l(\mathcal{G})$

Overview

	Comp. Cost	$\|A\|=1$	$\|A\|>1$	Duplicates
Naive - Enumerate all DAGs:				
global IDA (Maathuis et al, 2009)	$\mathcal{O}(\|V\|!)$	✓	-	Yes
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Enumerate valid parent sets of A:				
local IDA (Maathuis et al, 2009, Fang & He, 2020)	$\mathcal{O}(2^{l(\mathcal{G})})$	✓	-	Yes
semi-local IDA, joint IDA (P. et al, 2017,Nandy et al, 2017)	$\mathcal{O}(2^{l(\mathcal{G})} poly(\|V\|))$	✓	✓	Yes
optimal IDA (Witte et al, 2020)	$\mathcal{O}(2^{l(\mathcal{G})} poly(\|V\|))$	✓	$\sim$	No
Enum. $A-$ on poss. causal paths to Y:				
collapsible IDA (Liu et. al, 2020)	$\mathcal{O}((\|V\|+\|E\|)2^{r(\mathcal{G})})$	✓	-	Yes
Recursively enum. over shortest problem paths				
`IDGraphs` (Guo & Perković)	$\mathcal{O}(2^{m(\mathcal{G})} poly(\|V\|))$	✓	✓	No

- $l(\mathcal{G})$ - # of undirected edges connected to A
- $r(\mathcal{G})$ - # of edges $A-$ on possibly causal paths to Y, $r(\mathcal{G}) \leq l(\mathcal{G})$
- $m(\mathcal{G})$ - # of recursively id. edges $A-$ on proper possibly causal paths to Y, $m(\mathcal{G}) \leq r(\mathcal{G})$

Average runtime simulation comparison